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The Weierstrass Transform on Some Spaces of Type S

Abstract

The main objective of this paper is to study the classical theory of the Weierstrass transform is extended to the Dirac delta function and some of their basic properties. The characterizations of the Weierstrass transform on the Gel'fand-Shilov spaces of type *S* is shown.

Keywords: Weierstrass transform, Fourier transform, Green's function, Dirac delta function.

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Introduction

This present paper, my contribution is to define the Weierstrass transform in the form of Dirac delta function and study some of their basic properties. The characterizations of the Weierstrass transform on the Gel'fand-Shilov spaces of type *S* is shown.

The conventional Weierstrass transform of a suitably restricted function $f(x)$ on the real axis R is defined by [5, p. 267] as:

$$W[f(x)] = \frac{1}{2\sqrt{\pi}} \int_{\mathrm{R}} f(x) \exp\left[-\frac{(t-x)^2}{4}\right] dx. \qquad (1)$$

The Green's function $G(t-x,k\xi)$ of the heat equation for the infinite interval is $G(t-x,k\xi) = \frac{1}{2\sqrt{\pi k\xi}} \exp\left[-\frac{(t-x)^2}{4k\xi}\right]$, where k is positive constant.

The limit of $G(t-x,k\xi)$ as $\xi \to 0+$ represent the Dirac delta function $\delta(t-x) = \lim_{\xi\to 0+} \frac{1}{2\sqrt{\pi k\xi}} \exp\left[-\frac{(t-x)^2}{4k\xi}\right]$.

So, in terms of Dirac delta function $\delta(t-x)$, the Weierstrass transform (1) can be rewritten as

$$W[f(x)] = \hat{f}(x,t) = \int_{\mathrm{R}} f(x)\delta(t-x)dx. \qquad (2)$$

Aim of the Study/ The problem / Objective of the Study

The main objective of this paper is to develop the classical theory of the Weierstrass transform is extended to the Dirac delta function and some of their basic properties. The characterizations of the Weierstrass transform on the Gel'fand-Shilov spaces of type *S*, that is, S_α, S^β and S_α^β is shown. The Weierstrass transform arises naturally in problems involving the heat equation for one dimensional flow. The Weierstrass transform of certain class of generalized functions which are duals of so-called testing function spaces $W_{a,b}$ and $W(\alpha,\beta)$.

Review of Literature

The Weierstrass transform arises naturally in problems involving the heat equation for one dimensional flow. The Weierstrass transform of certain class of generalized functions which are duals of so-called testing function spaces $W_{a,b}$ and $W(\alpha,\beta)$ introduced by Zemanian [6]. The inversion formulas were also obtained. On the other hand the Weierstrass transform of bounded functions and certain other functions with prescribed growth conditions are all characterized by Hirshman and Widder [3]. Karunakaran and Vnugopal [4] introduced the Weierstrass transform on the generalized function $\mathfrak{I}$, obtained an inversion formula, and also characterize the Weierstrass transform of elements from both the testing function spaces $\mathfrak{I}$ and $\mathfrak{I}'$. In this present paper, motivated by the work of

Zemanian [6], and start with the Gel'fand-Shilov spaces of type *S*, that is, S_α, S^β and S_α^β defined by Gel'fand and Shilov [2] and develop some characterization of Weierstrass transform on these spaces.

Properties of the Weierstrass Transform
Relation between the Fourier transform and the Weierstrass Transform

The Fourier transform of $f(x)$ is defined by

$$F[f(x)](y) = \tilde{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathrm{R}} e^{-iyx} f(x)dx, \quad (3)$$

where the kernel e^{-iyx} of the Fourier transform can be written as $e^{-iyx} = \int_{\mathrm{R}} e^{-it}\delta(t-y)dt$.

So that,equn.(3) becomes

$$\tilde{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathrm{R}}\int_{\mathrm{R}} e^{-it}\delta(t-y) f(x)dxdt.$$

Substituting $y = sx$ and $t = sp$ where s is real, in the above expression, we obtain

$$\tilde{f}(sx) = \frac{1}{\sqrt{2\pi}} \int_{\mathrm{R}} e^{-isp}\left(s\int_{\mathrm{R}} f(x)\delta(sp-sx)dx \right)dp$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathrm{R}} e^{-isp} \hat{f}(x,p)dp = F\left[\hat{f}(x,p)\right] = F\left[W(f)\right].$$

This means that $\tilde{f}$ is the Fourier transform of f, whereas $F[W(f)]$ is the Fourier transform of the Weierstrass transform of f. Thus

$$\hat{f}(x,p) = W[f(.)].$$

Theorem1 (Linearity)

If $f(x),\ g(x) \in L^1(\mathrm{R})$ and a, b are any scalars, then $W[af(x)+bg(x)] = aW[f(x)]+bW[g(x)]$

Proof

By definition, we have

$$W[af(x)+bg(x)] = \int_{\mathrm{R}} [af(x)+bg(x)]\delta(t-x)dx$$

$$= a\int_{\mathrm{R}} f(x)\delta(t-x)dx + b\int_{\mathrm{R}} g(x)\delta(t-x)dx$$

$$= aW[f(x)]+bW[g(x)].$$

Theorem2 (Shifting)

If $W[f(x)] = \hat{f}(t,x)$, then

$$W[f(x-a)] = p\hat{f}(t-a).$$

Proof

By definition, we have

$$W[f(x-a)] = \int_{\mathrm{R}} f(x-a)\delta(t-(x-a))dx.$$

Substituting $x-a=\xi$, then the right-hand side of the above expression becomes

$$= \int_{\mathrm{R}} f(\xi)\delta(t-\xi)d\xi$$

$$= \int_R f(\xi)\left(\frac{1}{2\pi}\int_R e^{i(t-\xi)u}du \right)d\xi$$

$$= \int_R e^{itu}\left[\frac{1}{2\pi}\int_R\int_R e^{-it}\delta(t-\xi)f(\xi)d\xi dt \right]du$$

$$= \int_R e^{-it}W[f(\xi)]\delta(t)dt$$

$$= pW[f(\xi)]\int_R e^{-itp}\delta(tp)dt$$

$$= p\hat{f}(t-a).$$

The Spaces of Type S

The spaces of type S, that is, S_α, S^β and S_α^β Play an important role in the theory of linear partial differential equations as an intermediate spaces between the spaces of the C^∞ and the analytic functions. The Fourier transform has been studied on the spaces of type S by Friedman [1] and Gel'fand and Shilov [2]. The aim of this paper is to study the characterizations of the Weierstrass transform (2) on these spaces. Let us recall the definitions of these spaces.

Definition 1

The space $S_\alpha,\ \alpha \geq 0,$ consists of all infinitely differentiable functions $\phi(x),\ -\infty < x < \infty$, satisfying the inequalities

$$\gamma_{k,q}(\phi) = \sup_{x\in\mathrm{R}} \left| x^k \left(\frac{d}{dx}\right)^q \phi(x) \right| \leq C_q A^k k^{k\alpha},\ k,q \in \mathrm{N}_0,\quad (4)$$

where the constants *A* and C_q depend on the function ϕ. For k=0, the expression $k^{k\alpha}$ is considered to be equal to 1.

Definition2

The space $S^\beta,\ \beta \geq 0,$ consists of all infinitely differentiable functions $\phi(x),\ -\infty < x < \infty$, satisfying the inequalities

$$\gamma_{k,q}(\phi) = \sup_{x\in\mathbb{R}} \left| x^k \left(\frac{d}{dx}\right)^q \phi(x) \right|$$

$$\leq C_k B^q q^{q\beta},\ k,q\in\mathbb{N}_0,\ (5)$$

where the constants $B$and C_k depend on the function ϕ.

Definition3

The space $S_\alpha^{\ \beta},\ \alpha,\ \beta\geq 0,$ consists of all infinitely differentiable functions $\phi(x),\ -\infty < x < \infty$, satisfying the inequalities

$$\gamma_{k,q}(\phi) = \sup_{x\in\mathbb{R}} \left| x^k \left(\frac{d}{dx}\right)^q \phi(x) \right|$$

$$\leq CA^k B^q k^{k\alpha} q^{q\beta},\ k,q\in\mathbb{N}_0,\ (6)$$

where the constants A, $B$and C depend on the function ϕ.

The spaces of type S are closely interrelated by means of the Fourier transform; namely, the formulae

$$\hat{S}_\alpha = S^\alpha,\ \hat{S}^\beta = S_\beta\ and\ \hat{S}_\alpha^{\ \beta} = S_\beta^{\ \alpha} \quad \text{hold.} \quad (7)$$

We shall make use of the following inequalities in our investigation:

$$\frac{q!}{(q-k)!} = k!\binom{q}{k} \leq k!\sum_{k=0}^{q}\binom{q}{k} = k!2^q, \quad (8)$$

and $(k+q)^{(k+q)\alpha} \leq k^{k\alpha} q^{q\alpha} e^{k\alpha} e^{q\alpha},\ \forall k,q\in\mathbb{N}_0$ (9)

Theorem 3

The Weierstrass transform in terms of Dirac delta function is defined by (2) is a well-defined mapping of S_α into S^α for $\alpha\geq 0.$

Proof

Let $k,q\in\mathbb{N}_0$, we obtain

$$x^k\left(\frac{d}{dx}\right)^q W\left[f(x)\right] = x^k\left(\frac{d}{dx}\right)^q \int_{\mathbb{R}} f(x)\delta(t-x)dx$$

$$= x^k\left(\frac{d}{dx}\right)^q \int_{\mathbb{R}} f(x)\left\{\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(t-x)y}dy\right\}dx$$

$$= \frac{1}{2\pi}x^k\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{d}{dx}\right)^q\left(f(x)e^{i(t-x)y}\right)dxdy$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\sum_{k=0}^{q}\binom{q}{k}\left(\frac{d}{dx}\right)^k f(x)\left[(-i)^{q-2k}\int_{\mathbb{R}}\left(\left(\frac{d}{dy}\right)^k e^{-ixy}\right)y^{(q-k)}e^{ity}dy\right]dx.$$

Using integration by parts, the right-hand side of the last expression can be estimated by

$$\frac{1}{2\pi}\int_{\mathbb{R}}\sum_{k=0}^{q}\binom{q}{k}\left(\frac{d}{dx}\right)^k f(x)\left[(i)^{3q-4k}\int_{\mathbb{R}} e^{-ixy}\sum_{m=0}^{k}\binom{k}{m}\frac{k!}{(k-m)!}(ity)^{(k-m)}e^{ity}dy\right]dx.$$

Therefore,

$$\left|x^k\left(\frac{d}{dx}\right)^q W\left[f(x)\right]\right| \leq \frac{1}{2\pi}\sum_{k=0}^{q}\binom{q}{k}\sum_{m=0}^{k}\binom{k}{m}m!2^k\sum_{u=0}^{k}\binom{k}{u}$$

$$\times \sup_{x\in\mathbb{R}}\left|x^u\left(\frac{d}{dx}\right)^k f(x)\right|\int_{\mathbb{R}}\frac{dx}{(1+|x|)^q}\int_{\mathbb{R}}\frac{dy}{(1+|y|)^q}.$$

Now, using the Definition of (4) and the property (8), we can derive the above expression as

$$\leq \frac{1}{2\pi}q!2^q\sum_{k=0}^{q}\binom{q}{k}\sum_{m=0}^{k}\binom{k}{m}\sum_{u=0}^{k}\binom{k}{u}C_k A^u u^{u\alpha}$$

$$\times 4\int_0^\infty\frac{dx}{(1+|x|)^q}\int_0^\infty\frac{dy}{(1+|y|)^q}.$$

The integral can be made convergent by choosing $q>1$, then the right-hand side of the above expression can be bounded by

$$\frac{2}{\pi}q!2^q q^{q\alpha}\sum_{k=0}^{q}\binom{q}{k}\sum_{m=0}^{k}\binom{k}{m}\sum_{u=0}^{k}\binom{k}{u}C_k A^u$$

$$\leq \frac{2}{\pi}q!2^q q^{q\alpha}\sum_{k=0}^{q}\binom{q}{k}\sum_{m=0}^{k}\binom{k}{m}C_k(1+A)^k$$

$$\leq \frac{2}{\pi}q!2^q q^{q\alpha}(1+A)^q\sum_{k=0}^{q}\binom{q}{k}C_k 2^k$$

$$\leq C_k^{'} q^{q\alpha}\left(2^3(1+A)\right)^q$$

$$\leq C_k^{'} B^q q^{q\alpha}.$$

This completes the proof of the theorem.

Theorem 4

The Weierstrass transform in terms of Dirac delta function is defined by (2) is a well-defined mapping of S^β into S_β for $\beta\geq 0.$

Theorem 5

The Weierstrass transform in terms of Dirac delta function is defined by (2) is a well-defined mapping of S_α^β into S_β^α for $\alpha,\ \beta\geq 0.$

The proof of the above theorems are similar as the proofof Theorem 3.

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